

Mathematical Modelling in School – Examples and Experiences

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The paper reports on a university series of seminars on mathematical modelling at school based on the cooperation of mathematicians, mathematics educators and the schools. In these seminars, modelling exercises were carried out in classes at the upper secondary level of secondary schools.

First the theoretical framework of modelling in schools is discussed, followed by the framework and the structure of the course. Then diverse modelling attempts by students to solve one modelling example are presented. Finally an evaluation of the series of seminars is presented.

Over the last decade, the significance of the topic of modelling and real world examples in mathematics education has increased enormously within the mathematics didactics discussion as well as in public debate. For Werner Blum, since the beginning of his career in the field of mathematics didactics, the consideration of real world context and modelling formed a necessary component of an adequate and comprehensive understanding of mathematics as well as modern mathematics education. In the following sections, modelling examples and experiences from a series of university seminars about modelling in schools conducted at the University of Hamburg (in co-operation with my colleagues Claus Peter Ortlieb and Jens Struckmeier from the Department of Mathematics) is reported. For the design of this seminar, my experiences of joint seminars with Werner Blum when I was a research assistant at the University of Kassel played a decisive role¹.

1 Theoretical framework for modelling in mathematics education

Currently, a generally accepted goal of mathematics teaching is the acquisition of competencies and the ability to apply mathematics in everyday life. The PISA study carried out by the OECD, stated that the goal of mathematics teaching should be that pupils acquire mathematical literacy as it is stated in detail:

„Mathematical literacy is defined in PISA as the capacity to identify, understand and engage in mathematics, and to make well-founded judgements about the role that mathematics plays in an individual’s current and future private life, occupational life, social life with peers and relatives, and life as a constructive, concerned and reflective citizen.” (OECD 2001, p. 22)

This perception of the objectives of mathematics teaching has an impact on the structuring of mathematics lessons. It is insufficient to simply impart competencies for applying mathe-

¹ This article is based on a report written by Torben Willander and Eike Rath with a supplement by Magdalena Kornella and Björn Schwarz about the various seminars carried out (Kaiser et al. 2004).

matics only within the framework of school curriculum. Instead more mathematics teaching should deal with examples from which

- students understand the relevance of mathematics in everyday life, in our environment and for the sciences,
- students acquire competencies that enable them to solve real mathematics problems including problems in everyday life, in our environment and in the sciences.

This demand for new ways of structuring mathematics teaching meets the goals for more reality-oriented mathematics education as postulated in many didactic positions since the middle or the end of the 20th century in publications like those of Werner Blum and others. It has been agreed that mathematics teaching should not be reduced to just reality based examples but that these should play a central role in education (for an overview see Kaiser-Meßmer 1986, Blum 1996 and Maaß 2004).

Besides the application of standard mathematical procedures (such as applying well-known algorithms) in real world context and real world contexts serving as illustrations of mathematical concepts (e.g. usage of debts for the introduction of negative numbers), modelling problems as reality based contextual examples are increasingly regarded as being important. Within this modelling approach complex extra-mathematical problems are worked on based on a model perception of the relation of real world and mathematics. A modelling process is done on the basis of the following ideal-typical procedure: A real world situation is the process' starting point. Then the situation is idealised (named (a) in figure 1), i.e. simplified or structured in order to get a real world model. Then this real world model is mathematised (b), i.e. translated into mathematics so that it leads to a mathematical model of the original situation. Mathematical considerations during the mathematical model produce mathematical results (c) which must be reinterpreted into the real situation (d). The adequacy of the results must be checked, i.e. validated. In the case of an unsatisfactory problem solution, which happens quite frequently in practice, this process must be iterated.

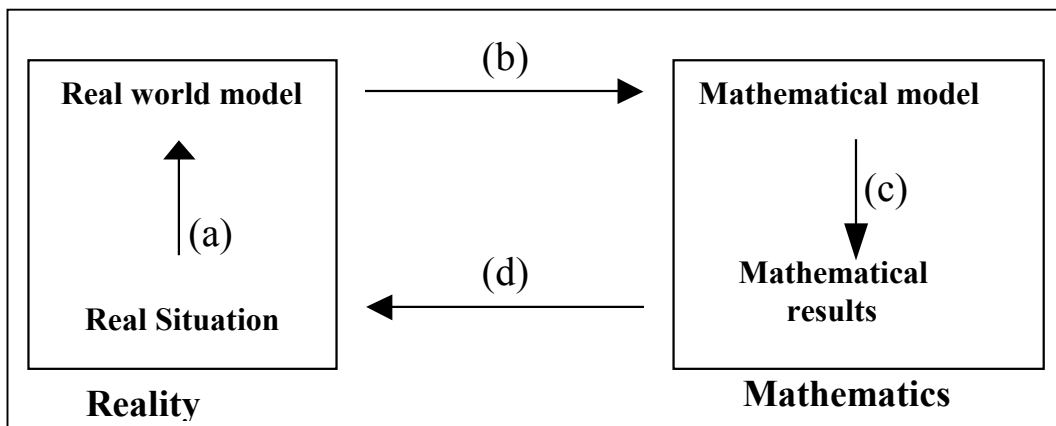


Fig. 1: Modelling process (from Kaiser 1995, p. 68 and Blum 1996, p. 18)

In applied mathematics, typically one does not distinguish a real model from a mathematical model, but regards the transition from real life situation into a mathematical problem as the core of modelling (see Ortlieb 2004, p. 7). Due to a lack of space, this differentiation will not be discussed further.

The competencies (or abilities) needed for this kind of modelling process are still the topic of current controversial debate. For instance, Maaß (2004, p. 35f), in her elaborate empirical study, gives a list of modelling competencies. In lieu of this study and based on my own unpublished research results, in my opinion the following competencies are needed:

- Competence to solve at least partly a real world based problem containing mathematics through a mathematical description (mathematical model) developed individually by one's own;
- Competence to reflect on the modelling process by activating meta-knowledge about modelling processes;
- Insight into the connections between mathematics and reality;
- Insight into the perception of mathematics as process and not merely as a product;
- Insight into the subjectivity of mathematical modelling, i.e. the dependence of modelling processes on the aims and the available mathematical tools and pupils competences;
- Social competences such as the ability to work in a group, and to communicate about and via mathematics.

This list is far from being complete since more extensive empirical studies are needed to receive well-founded knowledge about modelling competencies.

The following “recipes” were helpful for carrying out modelling examples and were discussed with the teacher students and the pupils:

- Formulate the real question or problem at the beginning as precisely as possible, clarify, which issues are relevant and which are irrelevant;
- Clarify the information needed to proceed: Is the information complete? Might the given information be useless or even misleading?
- After these two steps it makes sense to reflect upon the mathematical question to be treated and to formulate it precisely;
- Simplify the problem radically at the beginning, enlarge the model gradually if necessary;
- Check the mathematical solution found, whether it solves the real world problem; if not modify the model;
- Examine the model, whether it fulfils the criteria of admissibility, correctness and suitability, discuss the limitations of the model and assess it.

In order to promote a modelling understanding about mathematics and to develop competencies for carrying out modelling processes at school, it seems to be absolutely necessary to impart such competencies to prospective teachers during the course of their studies. In the following I will report about a university course with prospective teachers through which these students could acquire competencies for implementing modelling processes in their prospective teaching and through which their students could acquire competencies for carrying out modelling processes.

2 Framework and structure of the seminar

The project "Mathematical Modelling in School" was established in 2000 within the framework of the initiative "Mathematics at the Interface between School and University" financed by the Volkswagen Foundation and conducted by the Department of Mathematics in cooperation with the Didactics of Mathematics at the Department of Education at the University of Hamburg. This university course project with prospective teachers for upper secondary level teaching carried out every year with only one exception since year 2000, aims to establish a conjunction between university and school as well as between mathematics and didac-

tics of mathematics. Student groups supervised by the prospective teachers are the focus of the course. Each group works independently on one modelling example within the regular lessons or in separate after school working groups.

The main objective of the course is to change the academic curriculum of the Department of Mathematics and of the Didactics of Mathematics, so that in future mathematical modelling and associated teaching experiences will play a central role. Through this project, the prospective teachers will be enabled to implement modelling processes in mathematics teaching in their future professional work.

It was hoped that the participating students would acquire competencies to enable them to carry out modelling examples independently, i.e. the ability to extract mathematical questions from the given problem fields and to develop autonomously the solutions of real world problems. It is not the purpose of this project to provide a comprehensive overview about relevant fields of application of mathematics. Furthermore, it is hoped that students will be enabled to work purposefully on their own in open problem situations and will experience the feelings of uncertainty and insecurity, which are characteristics of real applications of mathematics in everyday life and sciences. An overarching goal is that students' experiences with mathematics and their mathematical world views or mathematical beliefs are broadened.

Each course extends over a period of two semesters with the following structure (in each cycle various modifications occurred; for details see Kaiser et al. 2004). After a short introduction into questions of teaching modelling, in a start-up lecture an authentic real life problem is presented by an applied mathematician. That is the problem which will be dealt with during more or less three months within the framework of school lessons. First, results will be presented by students at the end of the winter semester. During the summer semester a further real world modelling problem is worked on. Since modelling processes are carried out twice, both, the pupils and the attending prospective teachers, can review their experiences from the first run. Simultaneously a university course is taken where the students' solution attempts, problems and experiences are discussed.

Among others, until now the following modelling problems were treated:

- Mathematical methods within risk management
- Mathematics in private health insurance
- Mathematical and methodical problems of fishing sciences
- Optimal position of rescue helicopters in South Tyrol
- Radio-therapy planning for cancer patients
- Identification of fingerprints
- Pricing for internet booking of flights

Supplementary activities include excursions to companies for which mathematical models are of importance in order to demonstrate a broader variety of modelling examples. To give students an adequate imagination of the extensive applications of mathematical models, a series of lectures conducted by applied mathematicians is offered in which mathematical models from various fields of profession are presented at a level matching the students' knowledge.

In the following paragraph a modelling example will be described in detail in order to show the wide variety of solutions devised by the students.

3 Description of a modelling example

The example below was practiced at the Technical University of Kaiserslautern during the so called "Weeks of Modelling". The developed approaches to the problem were not known by

the participants of our modelling courses. During the winter semester 2001/2002 the same example was performed at a *Gymnasium* in Hamburg and in Norderstedt in an advanced mathematics course of year 12 (with 17 – 18 years-old). The students worked independently in groups of 4 and 5, and both, prospective teachers and teachers intervened only a little. Due to a lack of space the procedure of this example will only be demonstrated in a simplified, i.e. an ideal type, way, for details see Kaiser et al. (2004, p. 51ff).

Given was a skiing area (Fig. 2) for which the accident frequencies of various resorts were listed without specification of the referred time interval. As a help, first of all, the real world co-ordinates of the places were already transferred into a co-ordinate system (see Fig. 3). The co-ordinates of the resorts and the accident frequencies were handed to the pupils in the form of a table. (see Table 1).

Three rescue helicopters are provided by the relief organisation “the White Cross“ in this skiing area. The organisation makes a strong effort to help people who have had an accident as soon as possible.

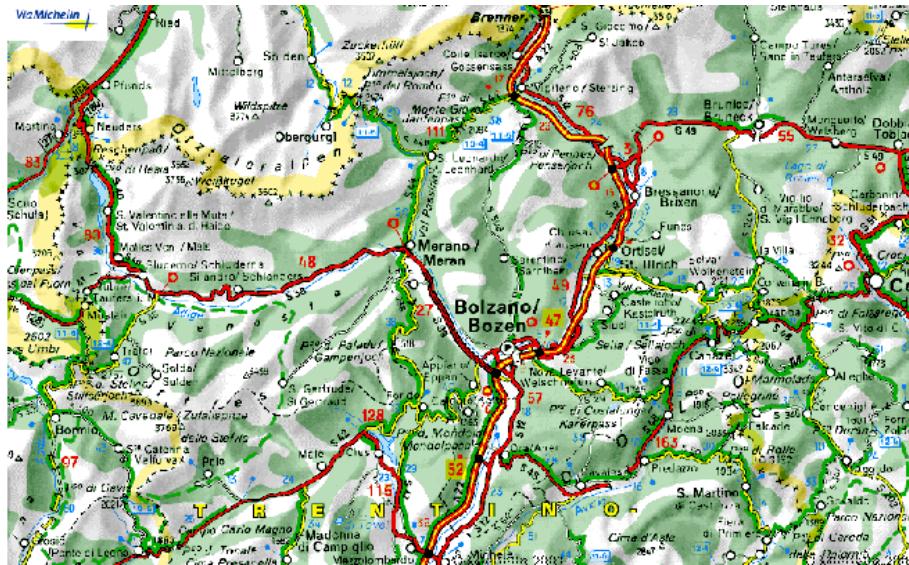


Fig. 2: Map of operational area

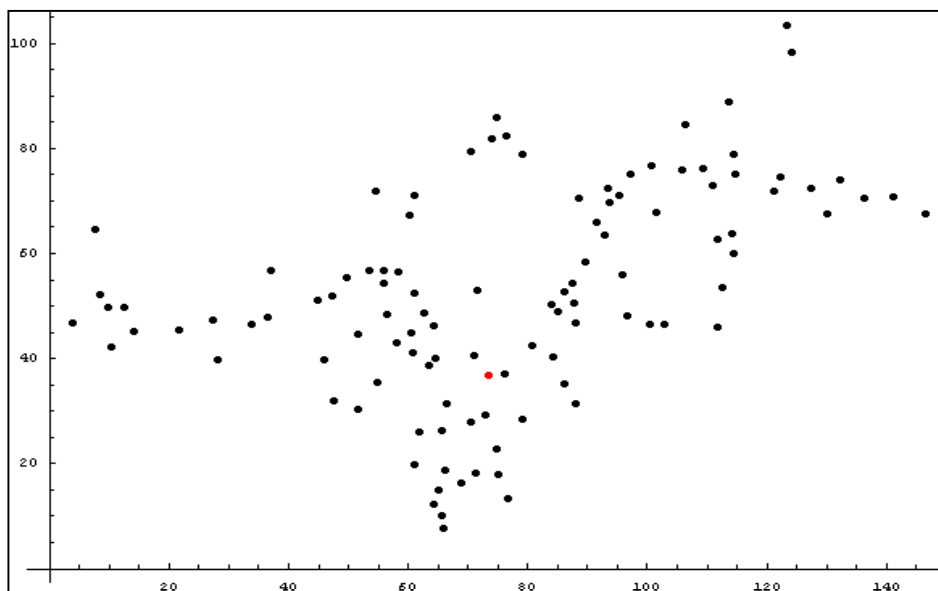


Fig. 3: Position of places where accidents happened

Orte	X	Y	N	Orte	X	Y	N	Orte	X	Y	N
Abtei	112,50	53,50	53	Lajen	87,75	50,50	11	Schenna	58,50	56,50	14
Ahrntal	124,25	98,25	26	Lana	56,50	48,25	4	Schlanders	27,50	47,25	21
Aldein	74,75	22,75	12	Latsch	34,00	46,50	8	Schluderns	12,50	49,75	1
Algund	53,50	56,75	3	Laurein	51,75	30,25	1	Schnals	37,00	56,75	40
Altrei	76,75	13,25	4	Leifers	73,00	29,25	5	Sexten	146,50	67,50	20
Andrian	63,50	38,50	2	Lüsen	101,50	67,75	11	St. Christina	100,50	46,50	50
Auer	61,25	19,75	15	Mals	8,50	52,00	20	St. Leonhard	61,00	71,00	16
Barbian	84,00	50,25	8	Margreid	64,25	12,25	7	St. Lorenzen	111,00	73,00	1
Bozen	73,50	36,75	96	Martell	28,25	39,75	7	St. Martin i, P.	60,25	67,25	19
Branzoll	62,00	26,00	4	Meran	56,00	54,25	32	St. Martin i, T.	111,75	62,50	12
Brenner	75,00	85,75	10	Mölten	64,25	46,25	14	St. Pankraz	51,75	44,50	13
Brixen	93,00	63,50	81	Montan	71,25	18,00	4	St. Ulrich	96,75	48,00	31
Bruneck	114,75	75,00	78	Moos	54,75	71,75	15	Sterzing	74,00	81,75	27
Corvara	111,75	46,00	62	Mühlbach	93,50	72,25	51	Stilfs	10,50	42,25	40
Deutschnofen	79,25	28,50	53	Mühlwald	106,50	84,50	3	Taufers	4,00	46,75	3
Enneberg	114,25	63,75	19	Nals	60,75	41,00	5	Terenten	100,75	76,75	6
Eppan	66,50	31,25	7	Naturns	45,00	51,00	9	Terlan	64,75	40,00	3
Feldthurns	89,75	58,25	8	Natz-Schabs	93,75	69,75	1	Tiers	86,25	35,25	24
Franzensfeste	88,75	70,50	3	Neumarkt	69,00	16,25	38	Tirol	56,00	56,75	2
Freienfeld	79,25	78,75	4	Olang	121,25	71,75	10	Tiesens	58,25	43,00	6
Gais	114,50	78,75	2	Partschins	49,75	55,25	8	Toblach	136,25	70,50	8
Gargazon	60,50	44,75	5	Pfalzen	109,50	76,00	2	Tramin	66,25	18,75	10
Glurns	10,00	49,75	2	Pfatten	70,50	27,75	2	Truden	75,25	17,75	8
Graun	7,75	64,50	19	Pfritsch	76,50	82,25	7	ULF-St. Felix	55,00	35,50	5
Gsies	132,25	74,00	5	Plaus	47,50	51,75	1	Ulten	46,00	39,75	38
Hafling	61,25	52,25	21	Prad	14,25	45,00	1	Vahrn	91,50	65,75	3
Innichen	141,25	70,75	13	Prags	130,25	67,50	8	Villanders	86,25	52,75	9
Jenesien	71,00	40,50	20	Prettau	123,50	103,25	7	Villnöß	96,00	56,00	18
Kaltern	65,75	26,25	30	Proveis	47,75	31,75	3	Vintl	97,25	75,00	13
Karneid	76,25	37,00	12	R.-Antholz	122,25	74,50	10	Völs	84,25	40,25	21
Kastelbell	36,50	47,75	3	Ratschings	70,50	79,25	38	Vöran	62,75	48,50	1
Kastelruth	88,00	46,75	76	Ritten	80,75	42,50	69	Waidbruck	85,25	49,00	4
Kiens	106,00	75,75	5	Rodeneck	95,25	71,00	7	Welsberg	127,50	72,25	3
Klausen	87,50	54,25	22	Salurn	66,00	7,50	12	Welschnofen	88,00	31,25	36
Kurtatsch	65,25	15,00	4	Sand	113,75	88,75	29	Wengen	114,50	60,00	4
Kurtinig	65,75	10,00	1	Sarntal	71,75	53,00	42	Wolkenstein	103,00	46,50	107
Laas	21,75	45,50	1								

Table 1: Coordinates and accident frequencies in skiing resorts

The modelling problem was to place the helicopters at the optimal position. For this, the students' main task was to define mathematically precisely what exactly is understood by an "optimal" positioning of the helicopters, in order to develop an assignment of each helicopters' locations of operation.

The first step was the transition into the real model for which pupils developed several definitions about what an optimal positioning means in connection with various criteria, for instance:

- The fastest possible first aid: No injured person should wait longer than 10 minutes;
- Equal usage of capacities: The helicopters should fly the same number of operations by referring to the available data;
- Minimisation of the distances to the places of operation: By using the available data as the starting point the distances should be minimised.

Furthermore, as a simplifying measure on the level of the real world, geographic facts were neglected (e.g. heliports could be positioned anywhere, all flight routes were possible).

The second step was the transformation into a mathematical model, for which the area was divided into three parts with one helicopter for each. This was done by drawing parallels to the second axis or by circles. The first and the second step were not strictly separated which is typical with modelling processes because the development of a model is done mostly in interdependence with the available mathematical means (for this reason – as already mentioned –

applied mathematicians tend towards other modelling schemata; for a critical insight on this problem see also Maaß 2004, p. 289 ff).

As the third step a mathematical solution was developed for which, depending on a chosen definition of optimality, various solution attempts were developed. In most cases, the groups followed only one optimality criterion but often added further criteria through which, due to the complexity of the problem, they often created big mathematical problems. The following problem solutions came out:

- Claim for minimising the length of the ways: A geometrical solution was chosen, i.e. the situation was simplified to two places and one helicopter. Then the centre of gravity was defined for which the co-ordinates of the places of operation were weighted with the accident frequencies. After that the model was extended to further places.
- Claim for using the helicopters' capacities as equally as possible, i.e. equal sum of distances flown by each helicopter: Each place of accident was related to one helicopter for which the division was done strongly related to a east-west partition. For the operational areas received by that, the location of each related helicopter was defined by calculating the centre of gravity, whereas no weighing based on the accident frequencies was done because it was also the aim that the rescue helicopters should reach a place of operation as fast as possible.
- Claim for attending injured persons as fast as possible: The area was divided into three circular areas and each helicopter was placed in the centre of the related circle. The circles were ordered in such a way that the distance flown by the helicopter from the base to the place of operation became as short as possible. Overlapping regions should be served by less frequently used helicopters. On the whole, this solution was mainly a graphical one.
- Claim for offering a first aid as fast as possible in connection with equal capacity utilisation of the helicopters: The skiing area was divided into three operational areas by parallels to the second axis. In the following the resorts were changed in order to get similar accident frequencies for each helicopter. The bases within the areas were defined by calculating the centre of gravity.

The fourth step comprised the verification of the solutions obtained according to the original situation. The plausibility of the results was examined and then it was asked whether the model was compatible with the original intention of the task.

In the beginning, the students felt that too much was demanded of them because they felt very insecure about how to approach the problem. However, In contrast to that, at the end of the project they identified themselves strongly with their results and defended them with enthusiasm and convincingly in various public presentations.

4 Experiences

The three modelling courses carried out from 2001 to 2004 were evaluated intensively. 180 students from 10 schools in Hamburg and its surrounding and 32 prospective teachers participated in the project. At the beginning and the end of the second and third run of the project all participants were questioned, whereas during the first run – due to organisational reasons – the questioning could be done only in the middle of the run. 138 students and 22 prospective teachers answered an open questionnaire which contained 11 questions on the domain beliefs about mathematics and mathematics teaching (based on the typology developed by Grigutsch 1996), about application of mathematics in everyday life and in the sciences; additionally students were asked what they want to study what kind of profession they would like to choose.

The second questionnaire included additional questions about the evaluation of the modelling examples they had worked on. For the prospective teachers the questionnaire contained additional questions about the university course. The evaluation of the questionnaires was conducted in accordance to the method of “thematic encoding” by Flick (1999) in which the codes were thematically deduced from the research questions as well as received through empirical open encoding. Due to a lack of space, the results are not represented in detail here (but an unpublished report is available from the author), so that in this contribution I concentrate more on some central results which are exemplified by direct verbal statements by students.

The following central results were achieved:

Result 1:

It has been shown that complex and high standard modelling examples are feasible in schools. A large number of the students involved participated actively up to the end of the course despite the long time period needed for the examples and the complexity of the examples. Most of the students expressed satisfaction with the results and with the course.

This is exemplified in the following statements of the students taken from the evaluation, in which they assess the modelling examples positively:

That *“this project ... makes lessons more varied and more interesting”*.

“Yes, because of this, relation to reality has been demonstrated, lessons are more flexible to so that teamwork is also promoted.”

“I am happy since in my opinion we achieved much.”

Result 2:

The results of the evaluation make clear that complex modelling examples are not reserved for highly talented and high performing students. On the contrary they can be carried out by average pupils in ordinary schools. The students’ solutions achieved a remarkably high standard, if we consider the wide spread of abilities and achievements that are usual in average German classes.

Result 3:

A change in the mathematical beliefs of the students and the prospective teachers concerning mathematics and mathematics teaching could be detected:

Before the project had started, static views about mathematics were dominant for most of the students and prospective teachers. In the following, some exemplary statements are given:

“Mathematics is constant which means what one learns today, will still be valid tomorrow”. Mathematics means *“dealing with numbers”*, *“a subject where one must calculate a lot.”*

After the project was finished, opinions that showed a more application oriented view came up such as

“Mathematics is the basis of many aspects of life and many professions”.

Mathematics is relevant, *“because in my opinion mathematics is essential in our everyday life, since mathematics holds many things together and makes many things possible for us.”*

Particularly in connection with descriptions of what good mathematics teaching should look like, the demand to include modelling examples into ordinary mathematics teaching was emphasised.

Students wished, that mathematics lessons have “*real world contexts*” and should be “*near to everyday life*” and “*deal with real problems of everyday life*”.

Result 4:

The prospective teachers assessed the seminar even more positively than the pupils.

It was positively noted that “*own teaching experiences were enhanced*” and that “*modelling in school practice*” was learnt.

It is demanded that

“*practical training for prospective teachers should generally be more integrated into university study*”.

Result 5:

Altogether, the evaluation shows a positive judgement of the modelling course, but clearly differentiated judgements about single examples are given. The modelling examples are generally judged as being “*very near to reality*”, allowing interdisciplinary insights, discovering danger e.g. in the fields of biometry. Students were critical of some examples, particularly the one concerned with the identification of fingerprints because of the considerable usage of computers, especially the usage of MATLAB. Students said that “*problems with the programmers*” occurred.

Result 6:

The teamwork practised within the modelling course was evaluated positively. The students described that through teamwork they could better sustain the uncertainty which is characteristic of modelling activities. In addition a greater selection of solution attempts was achieved. The students said:

“*Working in groups is super. The other participant can give you hints about things you don't notice. By sharing the work one achieves the result more quickly.*”

The learning of new ways of working, such as teamwork, is praised as a positive effect of the project.

Result 7:

Although one aim of the project had been to shift the students' intentions for their further university study towards mathematics only a small shift could be detected. Nevertheless a tendency towards favouring technical studies could be observed. This might be due to the fact that the lectures given by the applied mathematicians together with the modelling examples studied had shown the great applicability of mathematics in various fields, which might have directed the interest of the students towards these topic areas.

The following conclusions can be drawn from the experiences described above. There is a consensus that while on the one hand mathematics teaching changed positively through modelling examples, on the other hand the lessons became much more challenging and time-consuming.

Furthermore, it became clear that especially the learning and teaching practice at university changed positively through this kind of seminar.

In summary, the positive reactions from all participants demonstrate that through these kinds of seminars a change of mathematics teaching towards a stronger consideration of mod-

elling and real world context is possible. Particularly by involving prospective teachers, changes can be expected.

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